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WATER WAVE RUN-UP ON A BEACH

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ABSTRACT

While investigating the Tsunami wave project, a numerical method was devised to solve the initial boundary value problem for the equations of the non-linear shallow water theory. The case of one horizontal dimension was considered in a domain bounded by a shore at one end, with the motion of the shoreline taken into account. In addition, this method enabled the incident wave to be introduced at the seaward end of the domain. The water was assumed to be at rest in the domain until the incident wave arrived, and the bottom was assumed to slope uniformly.

INTRODUCTION

Some of the most important and most interesting effects of water waves occur when the waves approach a shore and run-up on a beach. In particular the damage done by a tsunami is produced then. Nevertheless this aspect of water waves has not been analyzed very extensively from a theoretical viewpoint because of the mathematical difficulties involved. However, in recent years some analytical methods have been developed which can be applied to the run-up problem.[1] Therefore, we have applied them and have obtained various new results. But these results are still mainly limited to waves of small amplitude because they are based for the most part on the linear theory of water waves. To obtain results valid for waves of larger amplitudes we have turned to numerical solution of the relevant partial differential equations by the method of finite differences. The numerical calculations are performed with the aid of a high speed digital computer.

In the first part of this report, we shall describe our analytical results. The analysis underlying them is contained in Appendix 1 and some of it is also in [1]. The objective of this part is to determine the amplification and run-up distance of time harmonic waves of any period approaching shore in water of arbitrarily varying depth. Since the analysis is based on the linear theory, the results can be superposed to treat the run-up of waves of arbitrary wave form. The results are

summarized in section 6 and graphed in figures 2-4. An analysis based upon the nonlinear theory is employed to determine the range of validity of these results, at least for long wave lengths.

In the second part of this report, we shall consider the numerical solution of the run-up problem. Since we are especially interested in tsunamis, which are long waves which develop large amplitudes, we shall base our study on the nonlinear shallow water theory of wave motions. The long wave length of tsunamis makes the shallow water theory applicable, and the large amplitudes necessitate the use of the nonlinear form of this theory. To solve the equations of this theory numerically, we first limit the region under consideration to that between the shoreline and some fixed boundary point far from the shore. Then we devise a numerical procedure for solving the equations of motion within this region. We also develop special methods for treating the motion of the shoreline and the boundary conditions at the fixed boundary point. We start our integration with the water at rest and with waves incident from the boundary. We continue the integration until some time after the waves have reached the shore and moved the shoreline up the beach and back down again, or even until this motion has happened many times.

The numerical procedure is general since both the bottom profile and the incident wave form can be varied and transient effects can be determined. To check the procedure a uniformly sloping bottom with time harmonic incident waves was treated in detail. The amplification factor was determined for various beach angles, wave frequencies and wave amplitudes and compared with the analytical results of the first part. For this

purpose it was necessary to keep the incident amplitude so small that bore formation did not occur. Agreement between the analytical and numerical results was satisfactory only for low frequency waves. The disagreement is under study and further calculations seem to indicate that better accuracy is required in the numerical scheme. Smaller net spacing can be used for this purpose but it is more efficient to devise a higher order accurate difference method.

Further steps to be taken in developing the numerical procedure are: to extend it to include the formation of a bore, its motion toward shore and its arrival at the shoreline. The motion of a bore toward shore has been dealt with previously [2], so only the problems of bore formation and arrival at the shore remain to be solved. From the analytical point of view, methods exist to analyze the formation and motion of a bore as long as it is far from shore. Analytical methods must still be devised to deal with the arrival of a bore at the shoreline.

ANALYTICAL RESULTS

1. Linear theory of run-up

Let us define the run-up distance r as the maximum horizontal distance which the water moves shoreward beyond the undisturbed shoreline. According to the linear theory of water waves.

$$r = \eta_o \cot \alpha \qquad 1.$$

Here η_o is the maximum wave height at the undisturbed shoreline and α is the angle between the bottom and the horizontal at the shoreline, i.e. the beach angle. Although the linear theory possesses solutions in which η_o is infinite, we consider them to be irrelevant for our purpose, and we shall deal only with solutions for which η_o is finite. In finding η_o from the linear theory, we shall consider only time harmonic wave motions, since other motions can then be treated by superposition.

A time harmonic wave motion is characterized by an angular frequency ω, a wave height far from shore, which we denote by a, and a direction of propagation. We shall assume that the shoreline is straight, that the bottom is a cylinder with generators parallel to the shore and that the incident wavefront is parallel to the shoreline. Then the motion is two dimensional. We describe it in terms of rectangular coordinates with origin at the undisturbed shoreline, the positive x-axis being normal to the shoreline and lying in the undisturbed water surface and the positive y-axis pointing vertically upward. Then the undisturbed water surface is the half line $y=o$, $x>o$. In these coordinates the equation of the bottom may be written as

$$y = -H(x) \qquad 2.$$

Since the shoreline is at $x = o$, we must have $H(o) = o$ and $H'(o) = \tan\alpha$. We shall also assume that the depth $H(x)$ approaches a constant value far from shore, so that

$$\lim_{x \to \infty} H(x) = h \qquad 3.$$

The possibility that the depth is infinite far from shore is included since h may be infinite.

As a consequence of linearity η_o is proportioned to the incident amplitude a so we may write

$$\eta_o = Aa \qquad 4.$$

We shall call A the amplification factor, and shall attempt to determine it. Once A is known the run-up factor $R = \gamma/a$ is, from (1) and (4),

$$R = A \cot\alpha \qquad 5.$$

In order to treat non-harmonic incident waves, we must find in addition to A, the phase lag θ between the occurrence of wave height maxima at points far from shore and at the shore for harmonic waves. Then we can use Fourier analysis to find the amplification for non-harmonic waves.

2. Uniformly sloping bottom

An exact explicit solution of the wave amplification problem is known only for the uniformly sloping bottom, for which $H(x) = x \tan \alpha$. It was first found by E. Isaacson [3] in 1950. An account of his analysis as well as a description of most other work on this problem, is given by J. J. Stoker [4]. The incident wave height in the infinitely deep water far above shore is given by

$$\eta_i(x, t) = a \cos (\omega t + \beta x) \qquad 6.$$

Here $\beta = \omega^2/g$ where g is the acceleration of gravity. At the shoreline $x = o$ the wave height for the finite solution is

$$\eta(t) = (2\pi/\alpha)^{1/2} a \cos (\omega t - \frac{\pi^2}{8\alpha} + \frac{\pi}{4}) \qquad 7.$$

In obtaining (7) we have renormalized Isaacson's solution so that the incident wave is given by (6) and we have corrected a misprint in his equation (3.8) and the preceding equation. From (6) and (7) we find that the amplification factor and phase shift are given by

$$A = (2\pi/\alpha)^{1/2} \qquad 8.$$
$$\theta = \pi/4 - \pi^2/8 \qquad 9.$$

It is to be noted that both A and θ are independent of the frequency ω but depend upon the beach angle α.

3. Slowly varying depth

We now wish to determine the amplification factor and phase lag for bottom contour more general than the uniformly sloping one. For this purpose we shall employ approximate solutions of the equations of the linear theory. First we consider an approximation which is appropriate when the water depth H(x) is a slowly varying function of x, i.e. when $(\beta H)^{-1}\, dH/dx \ll 1$. Then the geometrical optics method 5 yields an approximate solution which is good everywhere except near the shore. In this vicinity the linear shallow water theory yields a valid approximation. By matching these two solutions we determine the entire motion and find

$$A = (2\pi/\alpha)^{1/2} \frac{(K_o \sinh^2 \beta h K_o + \beta h K_o)^{1/2}}{\cosh \beta h K_o} \qquad 10.$$

$$\theta = \frac{\pi}{4} - \beta \int_0^\infty [K(x) - K_o]\, dx \qquad 11.$$

In (11) K(x) is the positive real root of the equation

$$K \tanh \beta K H(x) = 1 \qquad 12.$$

The constant $K_o = K(\infty)$ is the root of (12) with H(x) replaced by $h = H(\infty)$.

The amplification factor A given by (10) depends upon the beach slope α and the depth at infinity h, but it does not otherwise depend upon the manner in which H(x) varies between the shore and infinity. However the phase lag θ does depend upon this variation. Both A and θ also depend upon the frequency ω. This dependence is best expressed by saying that A and θ depend upon the dimensionless quantity βh. As βh becomes large, A tends to $(2\pi/\alpha)^{1/2}$ the value given by (8) for the

uniformly sloping bottom. Since that value is exact, this is a check on the approximate result. Another check is the fact that the result (11) for θ coincides with the exact result (9) when $H(x) = x\tan\alpha$

When Bh is small, (10) and (11) yield

$$A \sim \left(\frac{2\pi}{\alpha}\right)^{1/2} (4\beta h)^{1/4}, \qquad \beta h \ll 1 \qquad 13.$$

$$\theta \sim \frac{\pi}{4} - \beta^{1/2} \int_0^\infty \left[H^{-1/2} - h^{1/2}\right] dx, \quad \beta h \ll 1 \quad 14.$$

The expression (10) would indicate that A vanishes when βh is zero, but this unrealistic conclusion is not correct. The results (10), (11), (13) and (14) are not valid when βh is small because then the condition that the depth be slowly varying is not satisfied. Therefore we must now determine A and θ when βh is small.

4. **Shallow water**

The condition that βh be small implies that the water depth is small compared to the wavelength. In that case the wave motion can be described by means of the linear shallow water theory. From that theory we find

$$A = 2 + 2\beta b + O(\beta^2) \qquad h \ll 1 \qquad 15.$$
$$\theta = O(\beta^{3/2}) \qquad h \ll 1 \qquad 16.$$

The constant b in (15) is given by

$$b = \int_0^\infty x \left[H^{-1} - h^{-1}\right] dx \qquad 17.$$

Since usually $H(x) \leq h$, it follows that b is positive. Therefore A increases with increasing βh from the value A=2 at $\beta h = 0$. The values A=2 and $\theta = 0$ occur in reflection of waves from a vertical wall, as we see by setting $\alpha = \pi/2$ in (8) and (9). Thus the results (15) and (16 show that for $\beta = 0$, a beach of any contour reflects like a vertical wall.

When βh is large, the linear shallow water theory yields, when α is small,

$$A \sim (2\pi/\alpha)^{1/2} \, (4\beta h)^{1/4} \qquad \beta h \gg 1 \qquad 18.$$
$$\theta \sim \frac{\pi}{4} - \beta^{1/2} \int_0^\infty \left[H^{-1/2} - h^{-1/2}\right] dx \qquad \beta h \gg 1 \qquad 19.$$

The results are the same as (13) and (14). Thus the linear shallow water theory yields results which for $\beta h \gg 1$, match with the results for slowly varying bottoms, when the latter are expanded for $\beta h \ll 1$. This is the kind of agreement which is always obtained when two different asymptotic expansions of a given function are compared in a region of common validity. This agreement is another check upon our results.

5. Shallow water - an example

In the preceding section results for A and θ were given based upon the linear shallow water theory, for the cases $\beta h \ll 1$ and $\beta h \gg 1$. No general result is known for intermediate values of βh. Therefore we shall now consider a particular bottom profile for which the equations of the linear shallow water theory can be solved explicitly for all values of βh. This bottom slopes uniformly at the small angle α from the shore down to the depth h and then it is horizontal. (See figure 1.) For it we find

$$A = 2\left[J_0^2(2\alpha^{-1}\sqrt{\beta h}) + J_1^2(2\alpha^{-1}\sqrt{\beta h})\right]^{-1/2} \qquad 20.$$

$$\theta = \alpha^{-1}\sqrt{\beta h} - \tan^{-1}\frac{J_1(2\alpha^{-1}\sqrt{\beta h})}{J_0(2\alpha^{-1}\sqrt{\beta h})} \qquad 21.$$

When these results are expanded for βh small they agree with (15) and (16), while for βh large they agree with (18) and (19). This agreement provides a further check on our results. Graphs of A versus βh based upon (20) are shown in figure 2 for several values of α.

6. Summary of analytical results

The results (10) and (15) determine A for practically any bottom profile for both large and small values of βh. Similarly (11) and (16) determine θ. The bottom must have a small beach angle and a finite slope everywhere. The range of βh not covered by these results is relatively small, since it is a range throughout which the linear shallow water theory is applicable. The example treated above indicates that this range lies between .05 and .15 for $\alpha = \pi/20$ and between .05 and .25 for $\alpha = \pi/10$. As α increases, the gap widens while as α decreases it shrinks. For a bottom with very large slopes, the gap may be rather large.

Our results indicate that A increases monotonically from the value 2 at $\beta h = 0$ to a maximum at $\beta h = 1$ and then steadily diminishes to the asymptotic value $(2\pi/\alpha)^{1/2}$ at $\beta h = \infty$. The maximum value of A is about 10% larger than the asymptotic value. Only a small portion of the rising part of the curve of A versus βh is not covered by our general results. Graphs of A versus βh for several values of α are shown in figure 2, based upon (10).

Since our results are based upon the linear theory of water waves, they are applicable only to waves of such small amplitude that even after amplification, their wave height is small compared to their wavelength and to the depth. An explicit criterion for the range of applicability of the results for βh small can be obtained by utilizing the nonlinear shallow

water theory near the shore line, where the wave height is greatest. That theory shows that our results (10) and (11) are correct for βh small provided that the incident amplitude a satisfies

$$\text{a) } \beta a < \frac{\alpha^2}{A}, \text{ or equivalently b) } \frac{a}{h} < \frac{\alpha^2}{\beta h A} = \frac{1}{\omega^2 A} \qquad 22.$$

Here A is given by (10). Graphs of the right side of (22b) as a function of ω are shown in figure 3 for several values of α. When (22) is violated a discontinuity or bore forms at some distance from shore and travels toward the shore. For large values of βh there should be corrections to our results as βa increases, even in the absence of bore formation.

NUMERICAL RESULTS

1. Formulation of general problems

We introduce dimensionless variables: t, time; x, length; h(x, t) and H(x), water depths; u(x, t), water velocity. These are related to the previous dimensional variables, which are now denoted as $\bar{t}$, $\bar{x}$, $\bar{h}$, etc. by the scaling:

$$\bar{x} = xL,\ \bar{t} = t\frac{L}{\sqrt{gD}},\ \bar{u} = u\sqrt{gD},$$
$$\bar{h} = hD,\ \bar{H} = HD. \qquad 23.$$

For convenience we choose the scales L and D such that the dimensionless undisturbed water depth, H(x), satisfies:

$$H(o) = 1 \ ,\quad H(1) = O. \qquad 24.$$

With this convention the water initially occupies $x \leq 1$ and a uniformly sloping beach is given by

$$H(x) = 1-x. \qquad 25.$$

In dimensional variables the slope of such a beach is then $\tan\alpha = D/L$. (Note that now figure 1 should be reversed and shifted.)

The nonlinear shallow water theory equations become, in the dimensionless variables (23),

$$h_t + (uh)_x = O \qquad 26.$$
$$u_t + (1/2\ u^2)_x = H_x - h_x \qquad 27.$$

If the shoreline location at time t is denoted by $\xi(t)$ then since the water depth is always zero there

$$h(\xi(t),\ t) = O. \qquad 28.$$

In addition the velocity of the shoreline must be equal to the water velocity at the shoreline and so

$$\frac{d\,\xi(t)}{dt} = U(\xi(t), t). \tag{29}$$

Equations (28) and (29) are the boundary conditions to be imposed on the moving shoreline.

To facilitate computations we consider only the motion in $0 \leq x \leq \xi(t)$ for $t > 0$. If the flow at $x=o$ is always subsonic this can be done by specifying an appropriate wave incident from $x < o$. Introducing the local round speed, $c = \sqrt{h}$, the equations of motion (26) and (27) can be put into the characteristic form:

$$(u + 2c)_t + (u + c)(u + 2c)_x = H_x \tag{30}$$

$$(u - 2c)_t + (u - c)(u - 2c)_x = H_x \tag{31}$$

Equation (30) determines $u + 2c$ along the positive characteristics, $dx/dt = u + c$, which, as t increases, enter $x > o$ from $x < o$. Thus $u + 2c$ may be specified on $x = o$. If we assume that the linear theory is valid for $x < o$ and that $H(x) = \text{const} = 1$ there we can write the general solution of the linearized shallow water equations as

$$\left.\begin{aligned} u &= F(t-x) + f(t+x) \\ h - 1 &= F(t-x) - f(t+x) \end{aligned}\right\} \tag{32}$$

In the linear theory we use $c = \sqrt{h} = 1 + (1/2)(h - 1)$ and thus

$$u(o, t) + 2c(o, t) = 2\left[1 + F(t)\right]. \tag{33}$$

We shall use (33) as the boundary condition imposed at $x=o$. The significance of the function $F(t)$ is clarified in (32); i.e. it is the incident part of the linearized wave motion for $x < o$.

Initially the water is taken to be at rest, that is

$$h(x, o) = o, \quad u(x, o) = o \text{ in } o \le x \le \xi(o) = 1. \qquad 34.$$

In summary the problem posed above is: to solve the nonlinear shallow water equations (26) and (27) in $o \le x \le \xi(t)$ for $t > o$ subject to the boundary conditions (28) and (29) on the unknown boundary $x = \xi(t)$, the boundary condition (33) on $x=o$ and the initial conditions (34). The motion of the shoreline is also to be determined. In this study we only consider motions in which bores are not present.

2. Numerical method

The numerical method is a modification of the one employed in [2]. We use uniform spatial net, $x_j = j\Delta x$, and a time net, $t_{k+1} = t_k + \Delta t_k$, determined by the stability and smoothness conditions to be imposed below.

If we denote by $\xi_k = \xi(t_k)$ the position of the shoreline at time t_k then we define $X_{s(k)}$ as the net point for which:

$$X_{s(k)} + \frac{\Delta x}{2} < \xi_k \leq X_{s(k)} + \frac{3\Delta x}{2}. \qquad 35.$$

The calculations for t_{k+1} are then naturally divided into three types at different net points, as follows:

A) interior points: $x_1 \leq x_j \leq x_{s(k)}$;

Bo) incident wave: x_o ;

B_s) shoreline motion: $x_{s(k)}$ and ξ_{k+1} .

In the interior we use the conservation difference forms of (26) and (27) employed in [2] . Thus with the net points denoted as in Fig 4a the flow quantities at interior points A)are computed by:

$$h(P) = \frac{h(L') + h(R')}{2} - \frac{\Delta t_k}{2\Delta x}\left[u(R')h(R') - u(L')h(L')\right] \qquad 36.$$

$$u(P) = \frac{u(L')+u(R')}{2} - \frac{\Delta t_k}{2\Delta x}\left\{\left[\frac{u^2(R')-u^2(L')}{2}\right] + \left[h(R')-h(L')\right]\right\} + \Delta t_k H'(x_j)$$

The stability of these difference equations is insured by requiring the Courant condition: $\Delta t_k \leq \min\limits_{P'} \left(\frac{\Delta x}{|u(P')| + \sqrt{h(P')}} \right)$.

From (33) we recall that u+2c is specified at the point P in the incident wave configuration of Figure 4b. By (31) it follows that u-2c is propagated along the negative characteristics, $dx/_{dt} = u-c$. Thus if $|u| < c$ a difference form of (31) can be used to compute u-2c at P in terms of known data. Specifically we write

$$a \equiv u-2c, \qquad b \equiv u-c$$

and center the x- and t- differences at the center of the net rectangle in Fig. 4b to get

$$a(P) = a(R') + \frac{\left(1+\frac{\Delta t_k}{\Delta x}\tilde{b}\right)\left[a(P')-a(R)\right] + 2\,\Delta t\ \ H'(\Delta x/_2)}{\left(1 - \frac{\Delta t_k}{\Delta x}\ \tilde{b}\right)} . \qquad 37.$$

Here $\tilde{b} = (1/2)(b(R) + b(P'))$ is the value of b used at the center. Combining this result with that in (33) we get finally:

$$h(P) = \frac{1}{16}\left[a(P) - 2\ \ (1 + F(t_{k+1}))\right]^2$$

$$u(P) = \frac{1}{2}\left[a(P) + 2\ (1+F(t_{k+1}))\right] \qquad 38.$$

In Fig. 4c the netpoints entering into the determination of the shoreline motion are indicated. The dotted line R' R represents the moving shoreline. The unknown quantities are u(P), h(P), u(R) and ξ(R). The equations of motion, (26) and (27) hold at P and R.

However (26) is identically satisfied as $x \to \xi$, i.e. at R, by virtue of (28). The fourth equation is furnished by (29). Using this we write (27) at $x = \xi$ in the form

$$\frac{d\,u(\xi(t),t)}{dt} = H_x(\xi(t)) - h_x(\xi(t),t). \qquad 39.$$

We difference equations (26) and (27) in an implicit form essentially centered at P while (29) and (39) are integrated from t_k to t_{k+1} by the trapezoidal rule. The resulting equations are implicit and nonlinear and are solved by iteration. The form and order in which they are used is:

$$\xi(R) = \xi(R') + \frac{\Delta t_k}{2}\left[u(R) + u(R')\right] \qquad 40.$$

$$h(P) = h(P') - \frac{\Delta t_k}{\delta + \xi(R) - x_s}\left[0 - u(L)h(L)\right] \qquad 41.$$

$$u(P) = u(P') - \frac{\Delta t_k}{\delta + \xi(R) - x_s}\left\{\left[\frac{u^2(R) - u^2(L)}{2}\right] + \left[0 - h(L)\right]\right\} + \Delta t_k H'(x_s) \qquad 42.$$

$$u(R) = u(R') + \frac{\Delta t_k}{2}\left[H'(\xi_k) + H'(\xi(R)) - h_x(R') - h_x(R)\right] \qquad 43.$$

In (43) the quantities $h_x(R')$ and $h_x(R)$ are computed by extrapolating, to R' and R, h-difference quotients at times t_k and t_{k+1}, respectively. The iterations start by taking $u(R) = u(R')$ in (40). The value of $\xi(R)$ thus computed is used in (41) and (42) to obtain h(P) and u(P). These iterates are then used in (43) to compute $h_x(R)$ and hence u(R) which completes one cycle. The iterations cease when the total change in magnitude of all four quantities is less than a specified tolerance.

It is assumed that the difference equatious (40) - (43) and (37) when adjoined to the system (36) do not alter the stability condition previously imposed This seems to be borne out by the computations which have been run in some problems for several thousands of time steps. The time step is also restricted so that the shoreline will in general not move more than half the spatial mesh width in one time step; that is in addition tc the Courant condition we impose

$$\Delta t_k \leq \frac{\Delta x}{2\,|u(R)|} \quad .$$

If at the end of a time step $\xi_{k+1} = \xi(R) > X_{s(k)+1} + \delta/2$, we then set $s(k+1) = s(k)+1$ and interpolate u and h between P and R to get data at the new net point, (x_{s+1}, t_{k+1}). This interpolation is done at most every other time step as a result of the above restriction on Δt. When the shoreline recedes a similar procedure is employed and net points are discarded.

3. Results of Calculations.

All of the calculations to be reported used the uniformly sloping bottom profile represented in (25). A series of problems was run in which the incident wave function in (33) was of the form

$$F(t) = a \sin \omega t. \tag{44.}$$

The amplitude, a, and frequency, ω, were varied in these problems. Each case was run until at least ten full period waves had reached the shoreline. Most significant transient effects on the motion of the shoreline seemed to be absent after about five or six periods.

Since in our dimensionless variables the beach angle is $\pi/4$ the water height at the beach is equal to the displacement of the shoreline, $\xi(t)$. If the maximum steady state positive shoreline displacement is denoted by ξ_o then the amplification was calculated to be

$$A = \xi_o / a \tag{45.}$$

A list of values of the amplifications thus obtained is presented in Table 1. In Fig. 5 these values are plotted against the theoretical values of equations (10) and (20) for the special beach angle α for which $\tan \alpha = 1/10$. The abscissa is $\omega = \sqrt{\beta h}/\alpha$ and we see that the agreement is fairly good in the range $o \leq \omega \leq 3.3$. However, the numerically determined amplification then falls off rapidly and for $\omega > 5$ no reasonable values could be determined. This loss of accuracy for $\omega > 5$ suggests that the calculations become less accurate as ω increases and may explain the observed discrepancy. Calculations with finer meshes and possibly higher order accuracy are being considered.

APPENDICES

1. Analysis.

A method for obtaining an accurate approximation to the velocity potential $\varphi(x, y)$ of a time harmonic motion in water of slowly varying depth is presented in [5]. Since the complex wave height $\eta(x) = i\omega g^{-1}\varphi(x, o)$, that method also yields η. In the two dimensional case it leads to the following expression for two waves going in opposite directions.

$$\eta(x) = (K \sinh^2\beta KH + \beta KH)^{-1/2} \cosh\beta KH \left\{A_+ \exp\left[i\beta\int_0^x K(x)dx\right] + A_- \exp\left[-i\beta\int_0^x K(x)dx\right]\right\} \quad 1.$$

Here A_+ and A_- are constants and the other quantities have been defined in the text. For x large, so that $H(x) \approx h$, we may rewrite (1) as

$$\eta(x) = (K_o \sinh^2\beta K_o h + \beta K_o h)^{-1/2} \cosh K_o h \left\{A_+ \exp\left[i\beta\int_0^\infty (K-K_o)dx\right] \exp\left[i\beta K_o x\right] + A_- \exp\left[-i\beta\int_0^\infty (K-K_o)dx\right] \exp\left[-i\beta K_o x\right]\right\} \quad 2.$$

$$= a \exp\left[-i\beta K_o x\right] + a_+ \exp\left[i\beta K_o x\right]$$

The constants a and a_+ are defined by this equation, which represents a wave of amplitude a approaching the shore at x=o and a reflected wave of amplitude a_+ traveling away from shore.

Near the shore $\beta H \ll 1$ and (12) in the text yields $K \sim (\beta H)^{1/2}$. Then (1) yields

$$\eta(x) = (4\beta H)^{-1/4} \left\{A_+ \exp\left[i\beta^{1/2}\int_0^x H^{-1/2}dx\right] + A_- \exp\left[-i\beta^{1/2}\int_0^x H^{-1/2}dx\right]\right\} \quad 3.$$

Since $H(x) = x \tan\alpha$ near shore, (3) becomes

$$\eta(x) = (4\beta x \tan\alpha)^{-1/4}\left\{A_+ \exp\left[i\left(\frac{4\ \ x}{\tan}\right)^{1/2}\right] + A_- \exp\left[-i\left(\frac{4\ x}{\tan}\right)^{1/2}\right]\right\} \qquad 4.$$

Both (3) and (4) yield an infinite value for η at x=o, but they are not valid there. This is because (1), and therefore (3) and (4), are asymptotic forms of η valid for βH large, but at the shore H vanishes. Therefore a different asymptotic form of η, which we may call a boundary layer expansion, is needed near the shore.

If α is small the linear shallow water theory can be used to find near the shore. When we set $H(x) = x\alpha$, since α is small, it yields

$$\eta(x) = \eta_0 \, J_0\left(\left[4\beta x/\alpha\right]^{1/2}\right) \qquad 5.$$

For large x, (5) becomes

$$\eta(x) \sim \frac{\eta_0}{\sqrt{2\pi}}\left(\frac{\alpha}{4\beta x}\right)^{1/4}\left\{\exp i\left[\left(\frac{4\ \ x}{\alpha}\right)^{1/2} - \frac{\pi}{4}\right] + \exp i\left[-\left(\frac{4\beta x}{\alpha}\right)^{1/2} + \frac{\pi}{4}\right]\right\} \qquad 6.$$

Upon comparing (4) with (6), we find that they become identical if $\tan\alpha$ is replaced by α and if A_+ and A_- are related to η_0 by

$$\eta_0 = A_+ (2\pi/\alpha)^{1/2} e^{i\pi/4} = A_- (2\pi/\alpha)^{1/2} e^{-i\pi/4} \qquad 7.$$

When (7) holds, the solution (5) valid near shore matches the solution (1) valid away from shore. Upon using (2) to express A_+ in terms of the incident amplitude a, we obtain from (7) the wave height at the shore

$$\eta_o = a\left(\frac{2\pi}{\alpha}\right)^{1/2} \frac{(K_o \sinh^2\beta K_o h + \beta K_o h)^{1/2}}{\cosh \beta K_{oh}}$$

$$\exp\left\{-i\left[\omega t + \frac{\pi}{4} - \beta \int_0^\infty (K-K_o)\,dx\right\}\right. \qquad 8.$$

The amplitude and phase of (8) yield the amplification A and phase lag θ given by (10) and (11) in the text.

When the water is shallow compared to the wavelength, which is the case when $\beta H(x) << 1$, the linear shallow water theory is applicable. In this theory $\eta(x)$ satisfies the equation

$$(H\eta_x)_x + \beta\eta = o\,, \qquad x \geq o \qquad 9.$$

An asymptotic solution of (9) for βH large and H_x small is given by (3), which is not valid near the shore. The solution (5) satisfies (9) near the shore. Upon matching these two solutions as in the last paragraph, we obtain (18) and (19) in the text. For small values of βH, we solve (9) by expressing $\eta(x)$ as a power series in β. The details are given in [1] and the results are contained in (15)-(17) in the text.

To treat the example of section 5 we use the linear shallow water theory (9). A solution of that equation in the constant depth region $x > h \cot\alpha$, where $H(x) = h$, is given by the last form of (2) above. A solution in the uniformly sloping region $o<x<h \cot\alpha$ is given by (5).

At the point $x=h \cot\alpha$ we require $\eta(x)$ and the horizontal velocity to be continuous, from which it follows that η_x is also continuous. Upon imposing these two continuity conditions on (2) and (5) we can determine both η_o and a_+ in terms of the incident amplitude a. For small α when $\cot\alpha \approx \alpha^{-1}$ the result for η_o is

$$\eta_o = \frac{2\, e^{-i\alpha^{-1}\sqrt{\beta h}}}{o(2\alpha^{-1}\sqrt{\beta h}) + iJ_o^{1}(2\alpha^{-1}\sqrt{\beta h})}\, a \qquad 10.$$

By taking the amplitude and phase of (10) and recalling that $J_o^{1} = - {}_1$, we obtain (20) and (21) in the text.

For small angles α the amplification A becomes large. Then the linear theory of water waves fails to be applicable unless the incident amplitude is small enough. To analyze the effect of larger incident amplitudes it is necessary to use a nonlinear theory, at least near shore where the wave height is greatest. Since the depth is small near shore, the nonlinear shallow water theory can be used there. Away from shore the linear theory can be used and the solutions given by the two theories can be matched together. For a uniformly sloping bottom G. F. Carrier and H. P. Greenspan [6] found a periodic solution of the nonlinear shallow water equations. If we replace their parameters a and l_o by $\eta_o = \alpha l_o\, a/4$ and se $= 4\alpha/\beta$, we can rewrite their solution far from shore in exactly the form (5) multiplied by $\cos\omega t$. The constant η_o still denotes the maximum wave height at the shore. Since their

solution is of exactly the form (5) far from shore, matching it to the linear theory yields exactly our previous results (10) and (11) in the text for A and θ. However, their solution is valid only when their constant $. < 1$. Since $a = 4\eta_o/\alpha l_o = \beta\eta_o\alpha^{-2}$, this condition becomes $\eta_o < \alpha^2/\beta^{-1}$. Upon setting $\eta_o = Aa$, this becomes $\beta a < \alpha^2 A^{-1}$ which is (22) of the text.

2. Figures.

Captions for Figures

Figure 1. A profile of a bottom which slopes from the shore at angle α down to the depth h, and then remains of constant depth.

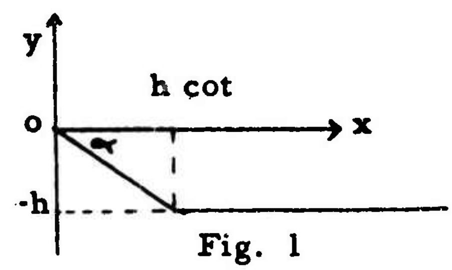


Fig. 1

Figure 2. Graphs of the amplification factor A as a function of βh for various values of the beach angle α. The curves labelled "A1" are based upon equation (10) and apply to any gently sloping bottom profile. The curves labelled "A2" are based upon equation (20) and apply only to the bottom profile of figure 1. For each small value of α, the corresponding curves "1" and "2" agree with each other over a region below h = .25. Curve "2" should be used to the left of this region and curve "1" to the right.

Figure 3. Graphs $(\omega^2 A)^{-1}$ as a function of ω for various values of the beach angle α. The curves are based upon equation (10) for A. According to equation (22), the amplification factor A is given by equation (10) only if $a/_h < (\omega^2 A)^{-1}$. When this condition is violated, the linear theory does not apply and bores will form. Thus these curves give the maximum value of $a/_h$, for each value of α and ω, for which figure 2 can be used.

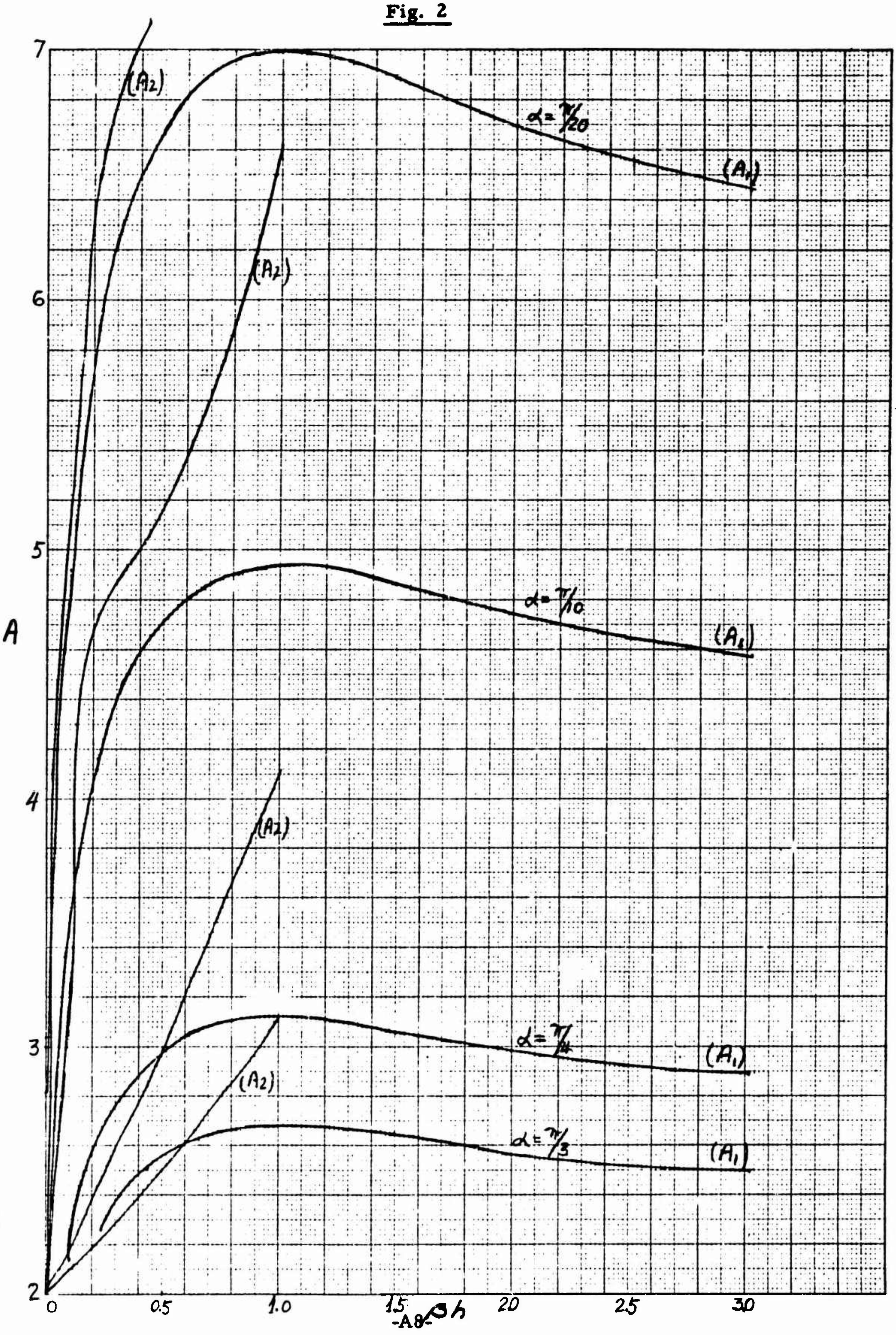
Figure 4a. Net point configuration for differencing the equation of motion.

4b. Net points entering into the incident wave boundary condition.

4c. Net points for determining the shoreline position.

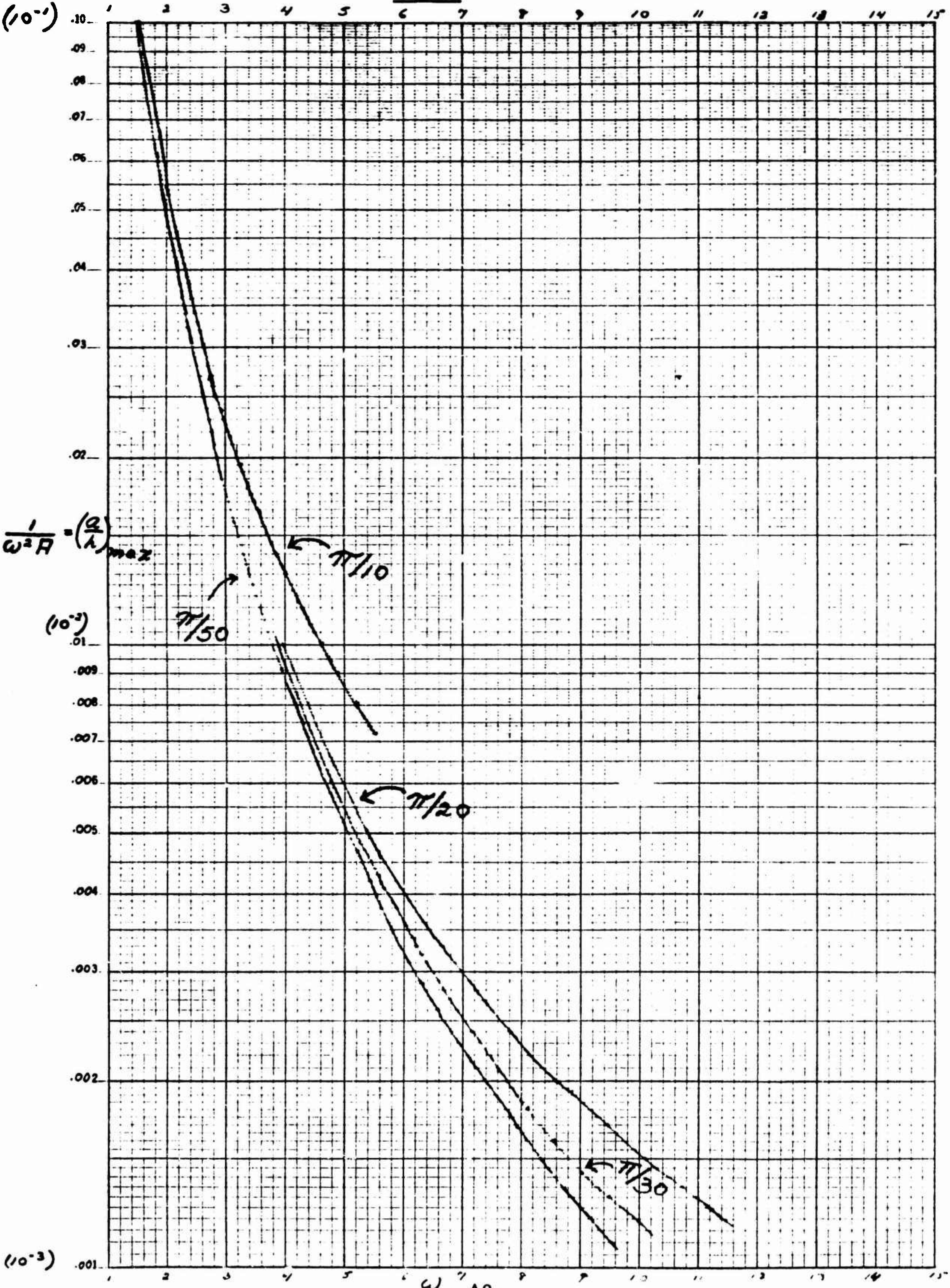
Figure 5. Comparison of numerically determined amplifications with the theoretical formulae (10), labelled A_1, and (20), labelled A_2, for a uniformly sloping beach with angle $\alpha = \tan^{-1} 1/10$. The numerical values are those from Table 1 for an incident amplitude $a/h = .004$.

Fig. 2



A
βh
7
6
5
4
3
2
0
0.5
1.0
1.5
2.0
2.5
3.0
(A_2)
$\alpha = \pi/20$
(A_1)
$\alpha = \pi/10$
$\alpha = \pi/4$
$\alpha = \pi/3$

Fig. 3



(10⁻¹)
.10
.09
.08
.07
.06
.05
.04
.03
.02
(10⁻²)
.01
.009
.008
.007
.006
.005
.004
.003
.002
(10⁻³)
.001
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$\frac{1}{\omega^2 A} = \left(\frac{a}{\lambda}\right)_{max}$
π/10
π/50
π/20
π/30
ω

Figs. 4a, b, c

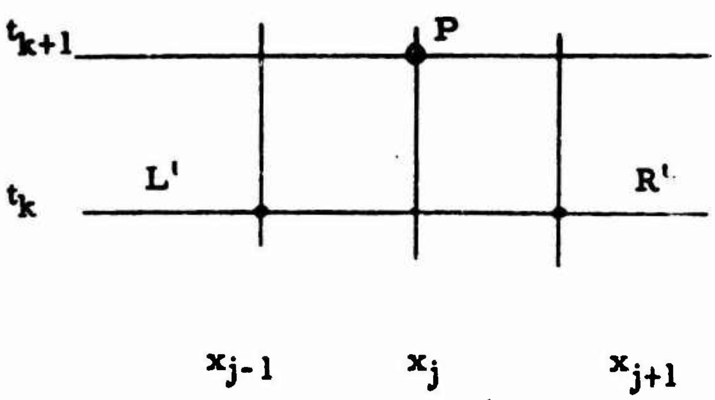


Fig. 4a

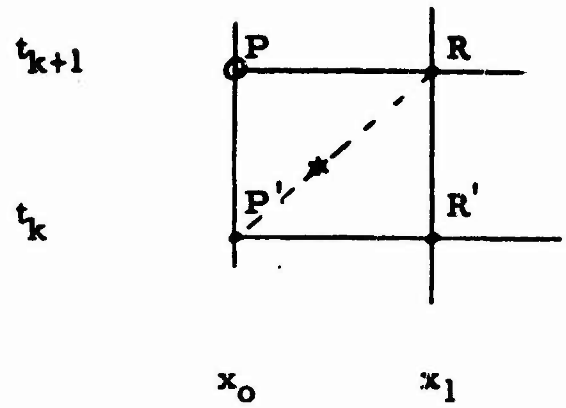


Fig. 4b

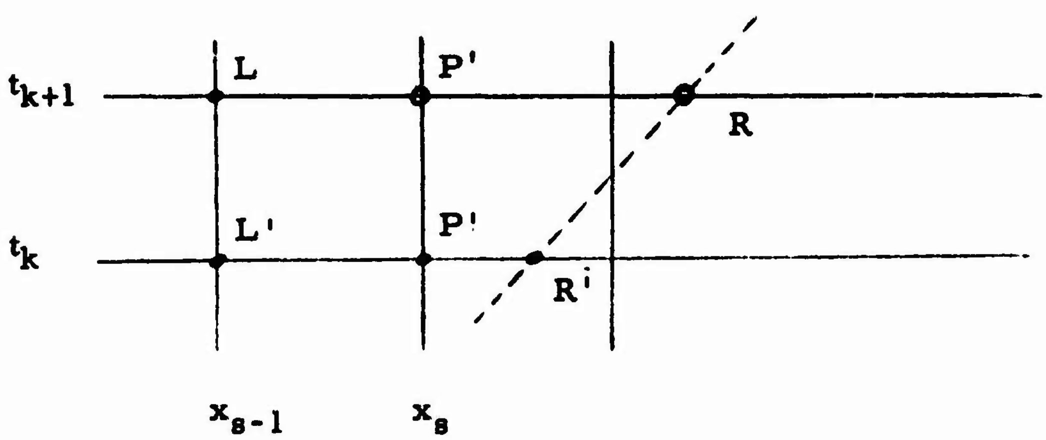
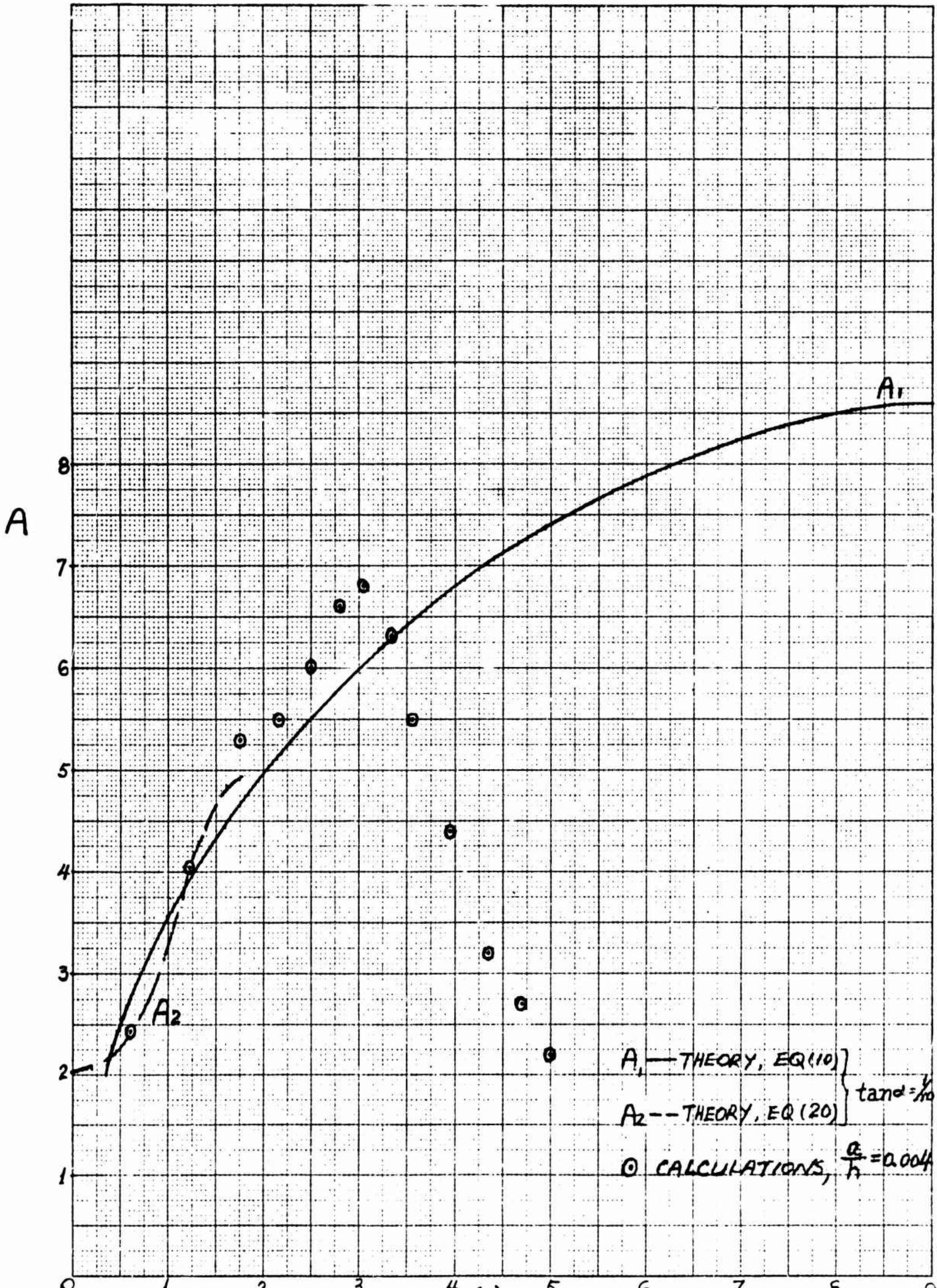


Fig. 4c

Fig. 5



A
ω
0
1
2
3
4
5
6
7
8
9
A1
A2
A1 — THEORY, EQ (10)
A2 - - THEORY, EQ (20)
tanα = 1/10
⊙ CALCULATIONS, a/h = 0.004

3. **Tables.**

Table 1

Computed amplification of an incident wave of the form $F(t) = a \sin \omega t$, for indicated values of ω and a.

		Incident amplitude, a		
	Frequency, ω	.002	.004	.008
1	0.625	2.5	2.4	2.4
2	1.250	4.1	4.1	4.1
3	1.768	5.4	5.3	5.3
4	2.165	5.5	5.5	5.5
5	2.500	6.0	6.0	6.0
6	2.795	6.6	6.6	6.6
7	3.062	6.7	6.8	6.8
8	3.307	6.1	6.3	6.5
9	3.535	5.4	5.5	5.7
10	3.953	4.3	4.4	4.4
11	4.330	3.4	3.2	3.8
12	4.677	2.7	2.7	2.8
13	5.000	?	2.2	2.4
14	5.303	?	1.9	1.5
15	5.728	?	1.4	1.5
16	6.124	?	?	1.2

4. References.

1. J. B. Keller, Tsunamis - water waves produced by earthquakes, Proceedings of the conference on tsunami hydrodynamics, Doax Cox ed., 1964.
2. H. B. Keller, D. A. Levine and G. B. Whitham, J. Fluid Mech. **7**, 302 (1960).
3. E. Isaacson, Comm. Pure Appl. Math. **3**, 11 (1950).
4. J. J. Stoker, Water Waves, Interscience, New York, 1956.
5. J. B. Keller, J. Fluid Mech. **4**, 607 (1958).
6. G. F. Carrier and H. P. Greenspan, J. Fluid Mech. **4**, 97 (1958).